TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 311, Number 1, January 1989

CORRIGENDUM TO "CONVEX SUBCONES OF THE CONTINGENT CONE IN NONSMOOTH CALCULUS AND OPTIMIZATION"

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In [1], general calculus rules are presented for the directional derivative and subgradient associated with any tangent cone having a short list of properties. Some special cases of these general formulae are then discussed. The purpose of this note is to correct two errors made by the author in the derivation of those special cases. The reader is referred to [1] for definitions of the relevant tangent cones and other notation and terminology.

The first of the two errors involves the tangent cone k^∞. The following example demonstrates that the calculus rules for k^∞ given in Theorems 5.1 and 5.4 of [1] are incorrect, even for locally Lipschitzian functions, without an additional hypothesis.

Example 1. Define $f_1: \mathbf{R} \to \mathbf{R}$ and $f_2: \mathbf{R} \to \mathbf{R}$ by

$$f_1(x) := \begin{cases} 0, & x \leq 0, \\ -2^{-n}, & 2^{-n-1} \leq x \leq 2^{-n}, \ n \text{ an odd integer}, \\ -3x + 2^{-n}, & 2^{-n-1} \leq x \leq 2^{-n}, \ n \text{ an even integer}, \end{cases}$$

and

$$f_2(x) := \begin{cases} 0, & x \leq 0, \\ -3x + 2^{-n}, & 2^{-n-1} \leq x \leq 2^{-n}, \ n \text{ an odd integer}, \\ -2^{-n}, & 2^{-n-1} \leq x \leq 2^{-n}, \ n \text{ an even integer}. \end{cases}$$

Since f_1 and f_2 are Lipschitzian, $\operatorname{dom} f_1^T(0;\circ) = \operatorname{dom} f_2^T(0;\circ) = \mathbf{R}$. One can calculate that

$$f_1^k(0;y) = f_2^k(0;y) = \min\{0, -y\},$$

while

$$(f_1 + f_2)^k(0;y) = \min\{0, -3y\}.$$

Thus

$$f_1^{k^\infty}(0;y) = f_2^{k^\infty}(0;y) = \max\{0, -y\}$$

Received by the editors July 29, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58C20.

and

$$(f_1 + f_2)^{k^\infty}(0; y) = \max\{0, -3y\}.$$

In this example,

$$(f_1 + f_2)^{k^\infty}(0; y) > f_1^{k^\infty}(0; y) + f_2^{k^\infty}(0; y) \quad \forall y < 0,$$

even though the hypotheses of Theorems 5.1 and 5.4 of [1] are satisfied.

The source of the error in Theorems 5.1 and 5.4 is an error in Proposition 2.4, which is valid for the cones P and K^∞ as stated but is not valid for k^∞. Correct calculus rules for k^∞ can be derived by means of Lemma 2.3. We now give correct versions of the parts of Theorems 5.1 and 5.4 involving k^∞:

Theorem 2. *Let $F: E \to E_1$ be strictly differentiable on some $N_\delta(x_0)$, $f_1: E \to \overline{\mathbf{R}}$ finite and Lipschitz-like at x_0, and $f_2: E_1 \to \overline{\mathbf{R}}$ finite and Lipschitz-like at $F(x_0)$. Assume that* (4.6) *of* [1] *holds, and that*

$$(1) \qquad (f_1 + f_2 \circ F)^k(x_0; \circ) \geq f_1^k(x_0; \circ) + f_2^k(F(x_0); \nabla F(x_0)(\circ)).$$

Then (5.2) *and* (5.4) *of* [1] *hold. Equality holds in* (5.2) *and* (5.4) *if in addition $f_1^{k^\infty}(x_0; \circ) = f_1^k(x_0; \circ)$ and $f_2^{k^\infty}(F(x_0); \nabla F(x_0)(\circ)) = f_2^k(F(x_0); \nabla F(x_0)(\circ))$.*

Proof. Define M, C and z_0 as in the proof of [1, Theorem 4.2]. By Lemma 2.3 and Theorem 4.2, it suffices to show that $k_{M(C)}(Mz_0) \subset M(k_C(z_0))$. This inclusion follows from (1), since

$$\begin{aligned} k_{M(C)}(Mz_0) &= \operatorname{epi}(f_1 + f_2 \circ F)^k(x_0; \circ) \\ &\subset \operatorname{epi}[f_1^k(x_0; \circ) + f_2^k(F(x_0); \nabla F(x_0)(\circ))] \quad \text{by (1)} \\ &\subset M(k_C(z_0)) \end{aligned}$$

as in the proof of Theorem 4.2. □

Remark. Condition (1) holds, in particular, in either of the following cases:

(a) The case in which f_1 and f_2 are indicator functions. This means that $A = A' = k^\infty$ can be used in Proposition 4.6.

(b) The case in which $f_1^k(x_0; \circ) = f_1^K(x_0; \circ)$. This equality holds, for example, if f_1 is locally Lipschitzian and directionally differentiable at x_0.

Theorem 3. *Let $f_i: E \to \overline{\mathbf{R}}$, $i = 1, \dots, n$, be finite and Lipschitz-like at x_0, and define $f := (f_1, \dots, f_n)$. Let $F: \mathbf{R}^n \to \overline{\mathbf{R}}$ be finite at $f(x_0)$, isotone on $N_\delta(x_0) \cup B$ for some $\delta > 0$, and l.s.c. Assume that* (4.14) *of* [1] *holds, that each $f_i^{k^\infty}(x_0; \circ)$ is proper, and that*

$$(2) \qquad (F \circ f)^k(x_0; \circ) \geq F^k(f(x_0); f_1^k(x_0; \circ), \dots, f_n^k(x_0; \circ)).$$

Then (5.11) *and* (5.13) *of* [1] *hold. Equality holds in* (5.11) *and* (5.13) *if $F^{k^\infty}(f(x_0); \circ) = F^k(f(x_0); \circ)$ and $f_i^{k^\infty}(x_0; \circ) = f_i^k(x_0; \circ)$ for each i.*

Proof. Condition (2) implies that $k_{M(C)}(Mz_0) \subset M(k_C(z_0))$ for M, C, and z_0 defined as in Theorem 4.9. By Lemma 2.3 and Theorem 4.9 of [1], (5.11) and (5.13) hold. □

An additional hypothesis is also needed in Proposition 5.6. We now take the opportunity to correct this omission.

Proposition 4. *Let $F: E \to E_1$ be strictly differentiable on some $N_\delta(x_0)$ and $f: E_1 \to \overline{\mathbf{R}}$ Lipschitz-like and finite at $F(x_0)$. Assume that* (5.19) *of* [1] *holds, and that*

$$\operatorname{dom} f^K(F(x_0);\circ) \subset \nabla F(x_0)E. \tag{3}$$

Then for all $y \in E$,

$$(f \circ F)^{K^\infty}(x_0;y) = f^{K^\infty}(F(x_0);\nabla F(x_0)y). \tag{4}$$

Moreover, (5.21) *of* [1] *holds.*

Proof. As stated in [1], (5.19) of [1] implies that

$$(f \circ F)^K(x_0;\circ) = f^K(F(x_0);\nabla F(x_0)(\circ)). \tag{5}$$

To prove (4), one must show that $(y, r) \in K^\infty(f \circ F, x_0)$ if and only if $(\nabla F(x_0)y, r) \in K^\infty(f, F(x_0))$. To this end, let $(y, r) \in K^\infty(f \circ F, x_0)$ and $(w, s) \in K(f, F(x_0))$. By (3), which was omitted in [1], $w = \nabla F(x_0)z$ for some $z \in E$. Now (5) implies that $(z, s) \in K(f \circ F, x_0)$, and so $(y+z, r+s) \in K(f \circ F, x_0)$. By (5) again, it follows that $(w, s)+(\nabla F(x_0)y, r) \in K(f, F(x_0))$, and hence $(\nabla F(x_0)y, r) \in K^\infty(f, F(x_0))$. Conversely, if $(\nabla F(x_0), y, r) \in K^\infty(f, F(x_0))$ and $(w, s) \in K(f \circ F, x_0)$, then (5) and the definition of K^∞ imply that $(\nabla F(x_0)(y + w), r + s) \in K(f, F(x_0))$, so that $(y, r) + (w, s) \in K(f \circ F, x_0)$ by (5). Thus $(y, r) \in K^\infty(f \circ F, x_0)$ and the proof is complete. □

Acknowledgment. I am grateful to Lionel Thibault for correcting the omission in Proposition 5.6 and pointing out the error in the proof of Proposition 2.4 of [1].

References

1. D. E. Ward, *Convex subcones of the contingent cone in nonsmooth calculus and optimization*, Trans. Amer. Math. Soc. **302** (1987), 661–682.

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